Formulas used in ladderline loss measurements

From the Telegrapher's Equations we can derive an expression for the input impedance of a line Zin, knowing the load impedance Zl, the length of line l, the characteristic impedance of the line Zo and the complex propagation coefficient “gamma”:

$$Zin = Zo.\frac{Zl + Zo.tanh(\gamma l)}{Zo + Zl.tanh(\gamma l)} \quad \text{where} \quad \gamma = \alpha + j\beta$$

For a short-circuit termination $Zl = 0$ so:

$$\boxed{Zinsc = Zo.tanh(\gamma l)} \qquad \textbf{(1)}$$

For an open-circuit termination $Zl = \infty$ so:

$$\boxed{Zinoc = \frac{Zo}{\tanh(\gamma l)}} \qquad \textbf{(2)}$$

---------------------------- Identities & expansions ----------------------------

$$\tanh(x+y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x).\tanh(y)} \qquad \text{(A)}$$

$$\tanh(x) = -jtan(jx) \qquad \text{(B)}$$

$$\tanh(jx) = -jtan(-x) \qquad \text{(C)}$$

$$\tanh(x) = \frac{1 - e^{-2x}}{1 + e^{-2x}} \qquad \text{(D)}$$

$$\tanh(x) = x - x^3/3 + 2x^5/15 ..... \qquad \text{(E)}$$

Short-circuit termination

Expanding equation (1) using identity (A) we get:

$$Zinsc = Zo.\frac{\tanh(\alpha l)+\tanh(j\beta l)}{1+\tanh(\alpha l).\tanh(j\beta l)}$$

then using identity C:

$$Zinsc = Zo.\frac{\tanh(\alpha l)-jtan(-\beta l)}{1-jtanh(\alpha l).\tan(-\beta l)}$$

If the line is an even number of quarter-waves long, $\tan(-\beta l)=0$ and Zin is resistive, so:

$$\boxed{\boldsymbol{Rin = Zo.tanh(\alpha l)}} \qquad \textbf{(3)}$$

If the line is an odd number of quarter-waves long $\tan(-\beta l)=\pm\infty$ so for either sign:

$$\boxed{\boldsymbol{Rin' = \frac{Zo}{\tanh(\alpha l)}}} \qquad \textbf{(4)}$$

where the “dash” distinguishes the high impedance case from the low impedance case.

Open-circuit termination

Expanding equation (2) with identity (A), then using (C), we get:

$$Zinoc = Zo.\frac{1-jtanh(\alpha l).\tan(-\beta l)}{\tanh(\alpha l)-jtan(-\beta l)}$$

If the line is an even number of quarter-waves long, $\tan(-\beta l)=0$ and Zin is resistive, so:

$$\boxed{\boldsymbol{Rin' = \frac{Zo}{\tanh(\alpha l)}}} \qquad \textbf{(5)}$$

If the line is an odd number of quarter-waves long $\tan(-\beta l)=\pm\infty$ so for either sign:

$$\boxed{\boldsymbol{Rin = Zo.tanh(\alpha l)}} \qquad \textbf{(6)}$$

where, again, the “dash” distinguishes the high impedance case from the low impedance case.

Determination of loss from Rin

Taking the low impedance cases - equation (3) or equation (6):

$$\boldsymbol{Rin = Zo.tanh(\alpha l)}$$

Then, using identity D: $Rin = Zo.\frac{1-e^{(-2\alpha l)}}{1+e^{(-2\alpha l)}}$

Cross multiplying: $Rin + Rin.e^{(-2\alpha l)} = Zo - Zo.e^{(-2\alpha l)}$

So: $e^{(-2\alpha l)} = \frac{Zo - Rin}{Zo + Rin}$

Now α is in Nepers/unit length, so $e^{(-\alpha l)}$ is the voltage attenuation along the line, and therefore $e^{(-2\alpha l)}$ is the power attenuation along the line. So loss in dB is simply given by:

$$\boldsymbol{Loss(dB) = -10.\mathrm{Log}\frac{Zo - Rin}{Zo + Rin}} \qquad \textbf{(7)}$$

Alternatively, using Expansion E for $\tanh(\alpha)$ and ignoring all but the first term in the series, for small losses we get the approximation:

$$Rin = Zo.tanh(\alpha l) \approx Zo.\alpha l$$

So: $Loss(Nepers) = \alpha l \approx Rin/Zo$

And: $\boldsymbol{Loss(dB) \approx 8.69.\ Rin/Zo}$

Alternatively, taking the high impedance cases - equation (4) or equation (5):

$$\boldsymbol{Rin'=\frac{Zo}{\tanh(\alpha l)}}$$

Then, using identity D: $$Rin'=Zo.\frac{1+e^{(-2\alpha l)}}{1-e^{(-2\alpha l)}}$$

Cross multiplying: $$Rin'-Rin'.e^{(-2\alpha l)}=Zo+Zo.e^{(-2\alpha l)}$$

So: $$e^{(-2\alpha l)}=\frac{Rin'-Zo}{Zo+Rin'}$$

Now α is in Nepers/unit length, so $e^{(-\alpha l)}$ is the voltage attenuation along the line, and therefore $e^{(-2\alpha l)}$ is the power attenuation along the line. So loss in dB is simply given by:

$$\boldsymbol{Loss(dB)=-10.\text{Log}\frac{Rin'-Zo}{Zo+Rin'}} \qquad \textbf{(8)}$$

Or, using Expansion E for $\tanh(\alpha)$ and ignoring all but the first term in the series, for small losses we get the approximation:

$$Rin'=\frac{Zo}{\tanh(\alpha l)}\approx\frac{Zo}{\alpha l}$$

So: $$Loss(Nepers)\approx\frac{Zo}{Rin'}$$

And: $$\boldsymbol{Loss(dB)\approx 8.69.\frac{(Zo)}{(Rin')}}$$

We can combine (7) and (8) into a more general expression that I attribute to Owen Duffy (ex-VK1OD):

$$\boldsymbol{Loss(dB)=-10.\text{Log}\frac{|Zo-R|}{Zo+R}} \qquad \textbf{(9)}$$

Note: typical values for Rin are within a range that can be measured relatively accurately by simple vector impedance meters ,whereas typical values for Rin' are not.

Determining Characteristic Impedance Zo from Zin

Rearranging equation **(1)**: $Zo=\frac{Zinsc}{\tanh(\gamma l)}$ (10)

Rearranging equation **(2)**: $Zo=Zinoc.tanh(\gamma l)$ (11)

Multiplying (10) by (11): $Zo^2=Zinoc.Zinsc$

Therefore, for any line length:

$$\boxed{Zo=\sqrt{Zinoc.Zinsc}} \quad \textbf{(12)}$$

Special case of line length = 1/8 wavelength or odd multiples

For shorted line: $Zin=Zo.tanh(\gamma l)$

Using identity (A): $Zin=Zo.\frac{\tanh(\alpha l)+\tanh(j\beta l)}{1+\tanh(\alpha l).\tanh(j\beta l)}$

And identity (C): $Zin=Zo.\frac{\tanh(\alpha l)-jtan(-\beta l)}{1-jtanh(\alpha l).\tan(-\beta l)}$

For 1/8 wavelength (and 5/8, 9/8 etc) $\tan(-\beta l)=-1$, so:

$$Zin=Zo.\frac{\tanh(\alpha l)+j1}{1+jtanh(\alpha l)}$$

Multiply top and bottom by conjugate $(1-jtanh(\alpha l))$ gives :

$$Zin=Zo.\frac{(\tanh(\alpha l)+j1)(1-jtanh(\alpha l))}{1+\tanh^2(\alpha l)}$$

If losses are small, square terms can be ignored, so:

$$Zin=Zo.\frac{\tanh(\alpha l)+j1+\tanh(\alpha l)}{1}$$

Then: $\boldsymbol{Imag(Zin) = Zo}$

and: $\boldsymbol{Real(Zin) = 2.Zo.\tanh(\alpha l)}$

For 3/8 wavelength (and 7/8, 11/8 etc) $\tan(-\beta l) = +1$, so:

$$Zin = Zo.\frac{\tanh(\alpha l) - j1}{1 - jtanh(\alpha l)}$$

Multiply top and bottom by conjugate $(1 + jtanh(\alpha l))$ gives:

$$Zin = Zo.\frac{(\tanh(\alpha l) - j1)(1 + jtanh(\alpha l))}{1 + \tanh^2(\alpha l)}$$

If losses are small, square terms can be ignored, so:

$$Zin = Zo.\frac{\tanh(\alpha l) - j1 + \tanh(\alpha l)}{1} \qquad (13)$$

Then: $\boldsymbol{Imag(Zin) = -Zo}$

and: $\boldsymbol{Real(Zin) = 2.Zo.\tanh(\alpha l)}$

So for 1/8 wavelength shorted line, and all odd multiples:

$$\boldsymbol{Zo = |Imag(Zin)|}$$

Note: for both types of termination $Real(Zin)$ is the same, and is equal to twice the value for a short-circuit half-wave line or an open-circuit quarter-wave line.

Following the same methodology for a 1/8 wavelength open-circuit line we again find that:

$$\boldsymbol{Real(Zin) = 2.Zo.\tanh(\alpha l)}$$

and: $\boldsymbol{Zo = |Imag(Zin)|}$

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